

# Universal cycles for permutations

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## Abstract

A universal cycle for permutations is a word of length  $n!$  such that each of the  $n!$  possible relative orders of  $n$  distinct integers occurs as a cyclic interval of the word. We show how to construct such a universal cycle in which only  $n + 1$  distinct integers are used. This is best possible and proves a conjecture of Chung, Diaconis and Graham.

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## 1. Introduction

A *de Bruijn cycle* of order  $n$  is a word in  $\{0, 1\}^{2^n}$  in which each  $n$ -tuple in  $\{0, 1\}^n$  appears exactly once as a cyclic interval (see [2]). The idea of a universal cycle generalizes the notion of a de Bruijn cycle.

Suppose that  $\mathcal{F}$  is a family of combinatorial objects with  $|\mathcal{F}| = N$ , each of which is represented (not necessarily in a unique way) by an  $n$ -tuple over some alphabet  $A$ . A *universal cycle* (or *ucycle*) for  $\mathcal{F}$  is a word  $u_1 u_2 \dots u_N$  with each  $F \in \mathcal{F}$  represented by exactly one  $u_{i+1} u_{i+2} \dots u_{i+n}$  where, here and throughout, index addition is interpreted modulo  $N$ . With this terminology a de Bruijn cycle is a ucycle for words of length  $n$  over  $\{0, 1\}$  with a word represented by itself. The definition of ucycle was introduced by Chung, Diaconis and Graham in [1]. Their paper, and the references therein, forms an good overview of the topic of universal cycles. The cases considered by them include  $\mathcal{F}$  being the set of permutations of an  $n$ -set,  $r$ -subsets of an  $n$ -set, and partitions of an  $n$ -set.

In this paper we will be concerned with ucycles for permutations: our family  $\mathcal{F}$  will be  $S_n$ , which we will regard as the set of all  $n$ -tuples of distinct elements of  $[n] = \{1, 2, \dots, n\}$ . It is not immediately obvious how we should represent permutations with words. The most natural thing to do would be to take  $A = [n]$  and represent a permutation by itself, but it is easily verified that (except when  $n \leq 2$ ) it is not possible to have a ucycle in this case. Indeed, if every cyclic interval of a word is to represent a permutation then our word must repeat with period  $n$ , and so only  $n$  distinct permutations can be represented. Another possibility which we mention in passing would be to represent the permutation  $a_1 a_2 \dots a_n$  by  $a_1 a_2 \dots a_{n-1}$ . It is clear that the permutation is determined by this. It was shown by Jackson [3] (using similar techniques to those used for de Bruijn cycles) that these ucycles exist for all  $n$ . Recently an efficient algorithm for constructing such ucycles was given by Williams [4]. He introduced the term *shorthand universal cycles for permutations* to describe them. Alternatively, Chung, Diaconis and Graham in [1] consider ucycles

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for permutations using a larger alphabet where each permutation is represented by any  $n$ -tuple in which the elements have the same relative order. Our aim is to prove their conjecture that such ucycles always exist when the alphabet is of size  $n + 1$ , the smallest possible. In contrast to the situation with shorthand universal cycles, the techniques used for de Bruijn cycles do not seem to help with this so a different approach is needed.

To describe the problem more formally we need the notion of order-isomorphism. If  $a = a_1a_2 \dots a_n$  and  $b = b_1b_2 \dots b_n$  are  $n$ -tuples of distinct integers, we say that  $a$  and  $b$  are *order-isomorphic* if

$$a_i < a_j \Leftrightarrow b_i < b_j$$

for all  $1 \leq i, j \leq n$ . Note that no two distinct permutations in  $S_n$  are order-isomorphic, and that any  $n$ -tuple of distinct integers is order-isomorphic to exactly one permutation in  $S_n$ . Hence, the set of  $n$ -tuples of distinct integers is partitioned into  $n!$  order-isomorphism classes which correspond to the elements of  $S_n$ .

We say that a word  $u_1u_2 \dots u_{n!}$  over an alphabet  $A \subset \mathbb{Z}$  is a ucycle for  $S_n$  if there is exactly one  $u_{i+1}u_{i+2} \dots u_{i+n}$  order-isomorphic to each permutation in  $S_n$ . For example 012032 is a ucycle for  $S_3$ . Let  $M(n)$  be the smallest integer  $m$  for which there is a ucycle for  $S_n$  with  $|A| = m$ . Note that if  $|A| = n$  then each permutation is represented by itself and so, as we noted earlier, no ucycle is possible (unless  $n \leq 2$ ). We deduce that  $M(n) \geq n + 1$  for all  $n \geq 3$ . Chung, Diaconis and Graham in [1] give the upper bound  $M(n) \leq 6n$  and conjecture that  $M(n) = n + 1$  for all  $n \geq 3$ . Our main result is that this conjecture is true.

**Theorem 1.** *For all  $n \geq 3$  there exists a word of length  $n!$  over the alphabet  $\{0, 1, 2, \dots, n\}$  such that each element of  $S_n$  is order-isomorphic to exactly one of the  $n!$  cyclic intervals of length  $n$ .*

We prove this by constructing such a word inductively. The details of our construction are in the next section. Having shown that such a word exists, it is natural to ask how many there are. In the final section we give some bounds on this.

Our construction works for  $n \geq 5$ . For smaller values of  $n$  it is a relatively simple matter to find such words by hand. For completeness examples are 012032 for  $n = 3$ , and 012301423042103421302143 for  $n = 4$ .

## 2. A construction of a universal cycle

We will show how to construct a word of length  $n!$  over the alphabet  $\{0, 1, 2, \dots, n\}$  such that for each  $a \in S_n$  there is a cyclic interval which is order-isomorphic to  $a$ .

Before describing the construction we make a few preliminary definitions.

As is standard for universal cycle problems we let  $G_n = (V, E)$ , the *transition graph*, be the directed graph with

$$V = \{(a_1a_2 \dots a_n) : a_i \in \{0, 1, 2, \dots, n\}, \text{ and } a_i \neq a_j \text{ for all } i \neq j\}$$

$$E = \{(a_1a_2 \dots a_n)(b_1b_2 \dots b_n) : a_{i+1} = b_i \text{ for all } 1 \leq i \leq n - 1\}.$$

Notice that every vertex of  $G_n$  has out-degree and in-degree both equal to 2.

The vertices on a directed cycle in  $G_n$  plainly correspond to the  $n$ -tuples which occur as cyclic intervals of some word. Our task, therefore, is to find a directed cycle in  $G_n$  of length  $n!$  such that for each  $a \in S_n$  there is some vertex of our cycle which is order-isomorphic to  $a$ . This is in contrast to many universal cycle problems where we seek a Hamilton cycle in the transition graph.

We define the map on the integers:

$$s_x(i) = \begin{cases} i & \text{if } i < x \\ i + 1 & \text{if } i \geq x. \end{cases}$$

We also, with a slight abuse of notation, write  $s_x$  for the map constructed by applying this map coordinatewise to an  $n$ -tuple. That is,

$$s_x(a_1a_2 \dots a_n) = s_x(a_1)s_x(a_2) \dots s_x(a_n).$$

The point of this definition is that if  $a = a_1a_2 \dots a_n \in S_n$  is a permutation of  $[n]$  and  $x \in [n + 1]$  then  $s_x(a)$  is the unique  $n$ -tuple of elements of  $[n + 1] \setminus \{x\}$  which is order-isomorphic to  $a$ . Note that, as will become clear, this is

the definition we need even though our final construction will produce a ucycle for permutations of  $[n]$  using alphabet  $\{0, 1, 2, \dots, n\}$ .

We also define a map  $r$  on  $n$ -tuples which permutes the elements of the  $n$ -tuple cyclically. That is,

$$r(a_1 a_2 \dots a_n) = a_2 a_3 \dots a_{n-1} a_n a_1.$$

Note that  $(a, r(a))$  is an edge of  $G_n$  and that  $r^n(a) = a$ .

As indicated above, we prove [Theorem 1](#) by constructing a cycle of length  $n!$  in  $G_n$  such that for each  $a \in S_n$  the cycle contains a vertex which is order-isomorphic to  $a$ . Our approach is to find a collection of short cycles in  $G_n$  which between them contain one vertex from each order-isomorphism class and to join them up. The joining up of the short cycles requires a slightly involved induction step which is where the main work lies.

**Proof of Theorem 1.** *Step 1:* Finding short cycles in  $G_n$ .

The first step is to find a collection of short cycles (each of length  $n$ ) in  $G_n$  which between them contain exactly one element from each order-isomorphism class of  $S_n$ . These cycles will use only  $n$  elements from the alphabet and we will think of each cycle as being “labelled” with the remaining unused element. Suppose that for each  $a = a_1 a_2 \dots a_{n-1} \in S_{n-1}$  we choose a label  $l(a)$  from  $[n]$ . Let  $0a$  be the  $n$ -tuple  $0a_1 a_2 \dots a_{n-1}$ . We have the following cycle in  $G_n$ :

$$s_{l(a)}(0a), r(s_{l(a)}(0a)), r^2(s_{l(a)}(0a)), \dots, r^{n-1}(s_{l(a)}(0a)).$$

We denote this cycle by  $\mathcal{C}(a, l(a))$ . As an example,  $\mathcal{C}(42135, 2)$  is the following cycle in  $G_6$ :

$$053146 \rightarrow 531460 \rightarrow 314605 \rightarrow 146053 \rightarrow 460531 \rightarrow 605314 \rightarrow 053146,$$

where arrows denote directed edges of  $G_6$ .

Note that for any choice of labels (that is any map  $l$ ) the cycles  $\mathcal{C}(a, l(a))$  and  $\mathcal{C}(b, l(b))$  are disjoint when  $a, b \in S_n$  are distinct. Consequently, whatever the choice of labels, the collection of cycles

$$\bigcup_{a \in S_{n-1}} \mathcal{C}(a, l(a))$$

is a disjoint union. It is easy to see that the vertices on these cycles contain between them exactly one  $n$ -tuple order-isomorphic to each permutation in  $S_n$ .

We must now show how, given a suitable labelling, we can join up these short cycles.

*Step 2:* Joining two of these cycles

Suppose that  $\mathcal{C}_1 = \mathcal{C}(a, x)$  and  $\mathcal{C}_2 = \mathcal{C}(b, y)$  are two of the cycles in  $G_n$  described above. What conditions on  $a, b$  and their labels  $x, y$  will allow us to join these cycles?

We may assume that  $x \leq y$ . Suppose further that  $1 \leq x \leq y - 2 \leq n - 1$ , and that  $a$  and  $b$  satisfy the following:

$$b_i = \begin{cases} a_i & \text{if } 1 \leq a_i \leq x - 1 \\ a_i + 1 & \text{if } x \leq a_i \leq y - 2 \\ x & \text{if } a_i = y - 1 \\ a_i & \text{if } y \leq a_i \leq n - 1. \end{cases}$$

If this happens we will say that the pair of cycles  $\mathcal{C}(a, x), \mathcal{C}(b, y)$  are *linkable*.

In this case  $s_x(0a)$  and  $s_y(0b)$  agree at all but one position; they differ only at the  $t$  for which  $a_t = y - 1$  and  $b_t = x$ . It follows that there is a directed edge in  $G_n$  from

$$r^t(s_x(0a)) = s_x(a_1 \dots a_{n-1} 0 a_1 \dots a_{t-1})$$

to

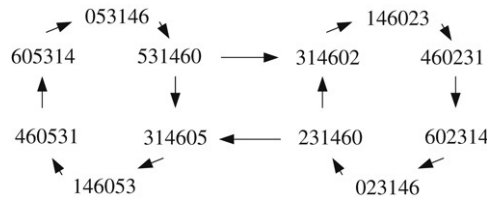
$$r^{t+1}(s_y(0b)) = s_y(b_{t+1} \dots b_{n-1} 0 b_1 \dots b_t).$$

Similarly, there is a directed edge in  $G_n$  from

$$r^t(s_y(0b)) = s_x(b_1 \dots b_{n-1} 0 b_1 \dots b_{t-1})$$

to

$$r^{t+1}(s_x(0a)) = s_y(a_{t+1} \dots a_{n-1} 0 a_1 \dots a_t).$$

Fig. 1. The cycles  $C(42135, 2)$  and  $C(23145, 5)$  are linkable.

If we add these edges to  $\mathcal{C}_1 \cup \mathcal{C}_2$  and remove the edges

$$r^t(s_x(0a))r^{t+1}(s_x(0a))$$

and

$$r^t(s_y(0b))r^{t+1}(s_y(0b)),$$

then we produce a single cycle of length  $2n$  whose vertices are precisely the vertices in  $\mathcal{C}_1 \cup \mathcal{C}_2$ .

We remark that if  $x = y - 1$  then the other conditions imply that  $a = b$  and so although we can perform a similar linking operation it is not useful. If  $x = y$  then  $b$  is not well defined.

As an example of the linking operation consider the linkable pair of 6-cycles  $C(42135, 2)$  and  $C(23145, 5)$  in  $G_6$ . If we add the edges  $531460 \rightarrow 314602$  and  $231460 \rightarrow 314605$ , and remove the edges  $531460 \rightarrow 314605$  and  $231460 \rightarrow 314602$  then a single cycle of length 12 in  $G_6$  is produced. These cycles and the linking operation are shown in Fig. 1.

**Step 3: Joining all of these cycles**

We now show that this linking operation can be used repeatedly to join a collection of disjoint short cycles, one for each  $a \in S_n$ , together.

Let  $H_n = (V, E)$  be the (undirected) graph with,

$$V = \{(a, x) : a \in S_{n-1}, x \in [n]\}$$

$$E = \{(a, x)(b, y) : \mathcal{C}(a, x), \mathcal{C}(b, y) \text{ are linkable}\}.$$

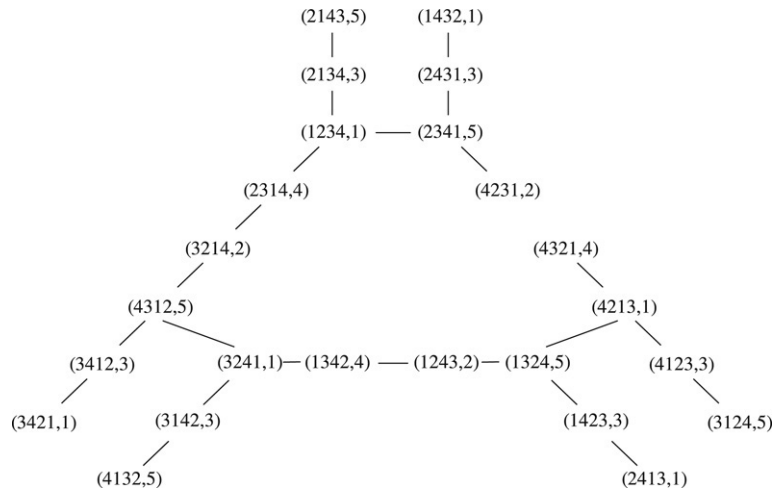
If we can find a subtree  $T_n$  of  $H_n$  of order  $(n-1)!$  which contains exactly one vertex  $(a, x)$  for each  $a \in S_{n-1}$  then we will be able to construct the required cycle. Take any vertex  $(a, l(a))$  of  $T_n$  and consider the cycle  $\mathcal{C}(a, l(a))$  associated with it. Consider also the cycles associated with all the neighbours in  $T_n$  of  $(a, l(a))$ . The linking operation described above can be used to join the cycles associated with these neighbours to  $\mathcal{C}(a, l(a))$ . This is because the definition of adjacency in  $H_n$  guarantees that we can join each of these cycles individually. Also, the fact that every vertex in  $G_n$  has out-degree 2 means that the joining happens at different places along the cycle. That is if  $(b, l(b))$  and  $(c, l(c))$  are distinct neighbours of  $(a, l(a))$  then the edge of  $\mathcal{C}(a, l(a))$  which must be deleted to join  $\mathcal{C}(b, l(b))$  to it is not the same as the one which must be deleted to join  $\mathcal{C}(c, l(c))$  to it. We conclude that we can join all of the relevant cycles to the cycle associated with  $(a, l(a))$ . The connectivity of  $T_n$  now implies that we can join all of the cycles associated with vertices of  $T_n$  into one cycle. This is plainly a cycle with the required properties.

The next step is to find such a subtree in  $H_n$ .

**Step 4: Constructing a Suitable Tree**

We will prove, by induction on  $n$ , the stronger statement that for all  $n \geq 5$ , there is a subtree  $T_n$  of  $H_n$  of order  $(n-1)!$  which satisfies:

1. for all  $a \in S_{n-1}$  there exists a unique  $x \in [n]$  such that  $(a, x) \in V(T_n)$ ,
2.  $(12 \dots (n-1), 1) \in V(T_n)$ ,
3.  $(23 \dots (k-1)1(k)(k+1) \dots (n-1), k) \in V(T_n)$  for all  $3 \leq k \leq n$ ,
4.  $(32145 \dots (n-1), 2) \in V(T_n)$ ,
5.  $(243156 \dots (n-1), 3) \in V(T_n)$ ,
6.  $v(31245 \dots (n-1))$  is a leaf in  $T_n$ ,
7.  $v(24135 \dots (n-1))$  is a leaf in  $T_n$ .

Fig. 2. A suitable choice for  $T_5$ .

Where, for a tree satisfying property 1, we denote the unique vertex in  $V(T_n)$  of the form  $(a, x)$  by  $v(a)$ .

For  $n = 5$  a suitable tree can be found. One such is given in Fig. 2.

Suppose that  $n \geq 5$  and that we have a subtree  $T_n$  of the graph  $H_n$  which satisfies the above conditions. We will use this to build a suitable subtree of  $H_{n+1}$ .

A key observation for our construction is that the map from  $V(H_n)$  to  $V(H_{n+1})$  obtained by replacing each vertex  $(a_1 a_2 \dots a_{n-1}, x) \in V(H_n)$  by  $(1(a_1 + 1)(a_2 + 1) \dots (a_{n-1} + 1), x + 1) \in V(H_{n+1})$  preserves adjacency. It follows that subgraphs of  $H_n$  are mapped into isomorphic copies in  $H_{n+1}$  by this map. Further, applying a fixed permutation to the coordinates of the  $n$ -tuple associated with each vertex of  $H_{n+1}$  gives an automorphism of  $H_{n+1}$  and so subgraphs of  $H_{n+1}$  are mapped into isomorphic copies.

We take  $n$  copies of  $T_n$ . These copies will be modified to form the building blocks for our subtree of  $H_{n+1}$  as follows.

In the first copy we replace each vertex  $(a_1 a_2 \dots a_{n-1}, x)$  by

$$(1(a_3 + 1)(a_1 + 1)(a_4 + 1)(a_2 + 1)(a_5 + 1)(a_6 + 1) \dots (a_{n-1} + 1), x + 1).$$

By the observation above this gives an isomorphic copy of  $T_n$  in  $H_{n+1}$ . We denote this tree by  $T_{n+1}^{(0)}$ .

In the next copy we replace each vertex  $(a_1 a_2 \dots a_{n-1}, x)$  by

$$((a_3 + 1)1(a_2 + 1)(a_1 + 1)(a_4 + 1)(a_5 + 1) \dots (a_{n-1} + 1), x + 1).$$

We denote this tree by  $T_{n+1}^{(1)}$ .

For all  $2 \leq k \leq n-1$ , we take a new copy of  $T_n$  and modify it as follows. We replace each vertex  $(a_1, a_2 \dots a_{n-1}, x)$  by

$$((a_k + 1)(a_1 + 1)(a_2 + 1) \dots (a_{k-1} + 1)1(a_{k+1} + 1) \dots (a_{n-1} + 1), x + 1).$$

We denote these trees by  $T_{n+1}^{(2)}, T_{n+1}^{(3)}, \dots, T_{n+1}^{(n-1)}$ .

As we mentioned this results in  $n$  subtrees of  $H_{n+1}$ . They are clearly disjoint because the position in which 1 appears in the first coordinate of each vertex is distinct for distinct trees. Let  $F_{n+1}$  be the  $n$  component subforest of  $H_{n+1}$  formed by taking the union of the trees  $T_{n+1}^{(k)}$  for  $0 \leq k \leq n-1$ .

It is also easy to see that for every  $a \in S_n$  there is a vertex in  $F_{n+1}$  of the form  $(a, x)$  for some  $x \in [n+1]$ . It remains to show that the  $n$  components can be joined up to form a single tree with the required properties.

Notice that it is a consequence of the construction of the  $T_{n+1}^{(k)}$  that if  $v(l_1 l_2 \dots l_{n-1})$  is a leaf in  $T_n$  then the following vertices are all leaves in  $F_{n+1}$ :

- $v(1(l_3 + 1)(l_1 + 1)(l_4 + 1)(l_2 + 1)(l_5 + 1)(l_6 + 1) \dots (l_{n-1} + 1))$
- $v((l_3 + 1)1(l_2 + 1)(l_1 + 1)(l_4 + 1)(l_5 + 1) \dots (l_{n-1} + 1))$
- $v((l_k + 1)(l_1 + 1)(l_2 + 1) \dots (l_{k-1} + 1)1(l_{k+1} + 1) \dots (l_{n-1} + 1))$  for  $2 \leq k \leq n-1$ .

We claim that  $v(12 \dots n)$  is a leaf in  $T_{n+1}^{(0)}$ , and hence a leaf in  $F_{n+1}$ . This follows from the fact that  $v(24135 \dots (n-1))$  is a leaf in  $T_n$  and the remark above. We delete this vertex from  $F_{n+1}$  to form a new forest.

The construction of the  $T_{n+1}^{(k-1)}$  and the fact that  $(23 \dots (k-1)1(k)(k+1) \dots (n-1), k) \in V(T_n)$  for all  $3 \leq k \leq n$  means that  $(23 \dots k1(k+1)(k+2) \dots n, k+1) \in V(F_{n+1})$  for all  $3 \leq k \leq n$ . Further, the construction of the  $T_{n+1}^{(1)}$  and the fact that  $(32145 \dots (n-1), 2) \in V(T_n)$  means that  $(2134 \dots n, 3) \in V(F_{n+1})$ .

We add to  $F_{n+1}$  a new vertex  $(123 \dots n, 1)$  (this replaces the deleted vertex from  $T_{n+1}^{(0)}$ ) and edges from this new vertex to  $(23 \dots k1(k+1)(k+2) \dots n, k+1)$  for all  $2 \leq k \leq n$ . The previous observation shows that all of these vertices are in  $F_{n+1}$ , and it is easy to check, using the definition of linkability, that the added edges are in  $H_{n+1}$ . This new forest has only two components.

Similarly, the inductive hypothesis that  $v(31245 \dots (n-1))$  is a leaf in  $T_n$  gives that  $v(342156 \dots n)$  is a leaf in  $F_{n+1}$  (using the case  $k = 3$  of the observation on leaves). We delete this leaf from the forest, replace it with a new vertex  $(342156 \dots n, 1)$ , and add edges from this new vertex to  $(341256 \dots n, 3)$  and  $(143256 \dots n, 4)$ . The construction of  $T_{n+1}^{(2)}$  and the fact that  $(32145 \dots (n-1), 2) \in V(T_n)$  ensures that the first of these vertices is in  $F_{n+1}$ . The construction of  $T_{n+1}^{(0)}$  and the fact that  $(243156 \dots (n-1), 3) \in V(T_n)$  ensures that the second of these vertices is in our modified  $T_{n+1}^{(0)}$ .

These modifications to  $F_{n+1}$  produce a forest of one component—that is a tree. Denote this tree by  $T_{n+1}$ . We will be done if we can show that  $T_{n+1}$  satisfies the properties demanded.

Plainly,  $T_{n+1}$  contains exactly one vertex of the form  $(a, x)$  for each  $a \in S_n$ . By construction  $(12 \dots n, 1)$ , and  $(23 \dots t1(t+1)(t+2) \dots n, t+1)$  are vertices of  $T_{n+1}$  for all  $2 \leq t \leq n$ . Hence the first three properties are satisfied.

The construction of  $T_{n+1}^{(2)}$  and the fact that  $(123 \dots (n-1), 1) \in V(T_n)$  ensures that  $(32145 \dots n, 2)$  is a vertex of  $T_{n+1}$ . The construction of  $T_{n+1}^{(3)}$  and the fact that  $(32145 \dots (n-1), 2) \in V(T_n)$  ensures that  $(243156 \dots n, 3)$  is a vertex of  $T_{n+1}$ . Hence properties 4 and 5 are satisfied.

Finally, the construction of  $T_{n+1}^{(1)}$  and the fact that  $v(31245 \dots (n-1))$  is a leaf in  $T_n$  ensures that  $v(31245 \dots n)$  is a leaf in  $F_{n+1}$ . The modifications which  $F_{n+1}$  undergoes do not change this and so it is a leaf in  $T_{n+1}$ . The construction of  $T_{n+1}^{(2)}$  and the fact that  $v(31245 \dots (n-1))$  is a leaf in  $T_n$  ensures that  $v(241356 \dots n)$  is a leaf in  $F_{n+1}$ . Again, the modifications to  $F_{n+1}$  do not change this and so this vertex is still a leaf in  $T_{n+1}$ . Hence properties 6 and 7 are satisfied.

We conclude that the tree  $T_{n+1}$  has the required properties. This completes the construction.  $\square$

### 3. Bounds on the number of universal cycles

Having constructed a ucycle for  $S_n$  over the alphabet  $\{0, 1, \dots, n\}$  it is natural to ask how many such ucycles exist. We will regard words which differ only by a cyclic permutation as the same so we normalize our universal cycles by insisting that the first  $n$  entries give a word order-isomorphic to  $12 \dots n$ . We denote by  $U(n)$  the number of words of length  $n!$  over the alphabet  $\{0, 1, 2, \dots, n\}$  which contain exactly one cyclic interval order-isomorphic to each permutation in  $S_n$  and for which the first  $n$  entries form a word which is order-isomorphic to  $12 \dots n$ . There is a natural upper bound which is essentially exponential in  $n!$  based on the fact that if we are writing down the word one letter at a time we have 2 choices for each letter. We can also show that there is enough choice in the construction of the previous section to prove a lower bound which is exponential in  $(n-1)!$ . It is slightly surprising that our construction gives a lower bound which is this large. However, the upper and lower bounds are still far apart and we have no idea where the true answer lies.

#### Theorem 2.

$$420^{\frac{(n-1)!}{24}} \leq U(n) \leq (n+1)2^{n!-n}.$$

**Proof.** Suppose we write down our universal cycle one letter at a time. We must start by writing down a word of length  $n$  which is order-isomorphic to  $12 \dots n$ ; there are  $n+1$  ways of doing this. For each of the next  $n! - n$  entries we must not choose any of the previous  $(n-1)$  entries (all of which are distinct) and so we have 2 choices for each entry. This gives the required upper bound.

Now for the lower bound. We will give a lower bound on the number of subtrees of  $H_n$  which satisfies the conditions of step 4 of the proof of Theorem 1. It can be checked that if a universal cycle comes from a subtree of  $H_n$  in the way

described then the tree is determined by the universal cycle. It follows that the number of such trees is a lower bound for  $U(n)$ .

Notice that if we have  $t_n$  such subtrees of  $H_n$  then we have at least  $t_n^n$  such subtrees of  $H_{n+1}$ . This is because in our construction we took  $n$  copies of  $T_n$  to build  $T_{n+1}$  from and each different set of choices yields a different tree. We conclude that the number of subtrees of  $H_n$  satisfying the conditions is at least

$$t_5^{5 \times 6 \times \cdots \times n-1} = t_5^{\frac{(n-1)!}{24}}.$$

Finally, we bound  $t_5$ . We modify the given  $T_5$  by adding edges from (1432, 1) to (2143, 5), from (3421, 1) to (4132, 5) and from (4231, 2) to (4321, 4). This graph is such that any of its spanning trees satisfies the properties for our  $T_5$ . It can be checked that this graph has 420 spanning trees. This gives the lower bound.  $\square$

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